BOUNDED APPROXIMATION BY POLYNOMIALS WHOSE ZEROS LIE ON A CIRCLE(1)

BY

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1. **Introduction.** Let C be a rectifiable Jordan curve with interior D. We say that a sequence of polynomials P_n converges boundedly to a function f in D, or f is boundedly approximated by P_n in D, if P_n converges to f throughout D and $\sup \{|P_n(z)| : z \in D\}$ is bounded as a function of n. A polynomial whose zeros lie on C will be called a C-polynomial. It is obvious that the limit function of a boundedly convergent sequence of C-polynomials in D is a bounded zero free holomorphic function in D, unless it is identically zero. In this paper, we will present a proof of the somewhat unexpected converse for the case when C is a circle, as announced in [1].

MAIN THEOREM. *Every bounded zero free holomorphic function in the open unit disc can be boundedly approximated there by polynomials whose zeros lie on the unit circumference.*

More generally, suppose that C is any rectifiable Jordan curve, so smooth that its parametric representation in terms of arc length has a Hölder continuous derivative. The methods developed in this paper can be extended and modified to prove the possibility of bounded approximation by C-polynomials of functions f, defined and zero free in clos D, such that the derivative f', relative to clos D, exists and is Hölder continuous throughout clos D. In particular, any function f holomorphic and zero free on clos D can be boundedly approximated by C-polynomials [1], [2]. However, for arbitrary Jordan curves C, the problem of bounded approximation by C-polynomials is open, even when f is holomorphic and zero free in clos D.

It should be mentioned that a weaker kind of approximation by C-polynomials was studied by G. R. MacLane [7]. He proved that if C is a rectifiable Jordan curve with interior D and f is holomorphic and zero free in D, then there exists a sequence of C-polynomials which converges to f uniformly on every compact subset of D. This result was later extended by J. Korevaar and his students [5], [6], [8] to other domains D. Very recently, Professor Korevaar and the author considered the case where C is the disjoint union of two or more Jordan curves [3]. It is interesting to note how the approximation problem by C-polynomials breaks down in some situations.

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Throughout the rest of this paper, C will denote the unit circle and D, the open unit disc.

2. **Approximation of $S_k(z)=\prod_{m=1}^k (1-ze^{-i\theta_m})^{\alpha_m}$.** We first construct a sequence of C-polynomials P_n which converges to the function

$$S_k(z) = \prod_{m=1}^{k} (1-ze^{-i\theta_m})^{\alpha_m} \tag{2.1}$$

in D, where $0\leqq\theta_1<\cdots<\theta_k<2\pi$ and $0\leqq\alpha_m<1$, $m=1,\ldots,k$.

The proof of the uniform boundedness of P_n on D will be included in §4. We use the same construction as indicated in [1]. Set $\sum_{m=1}^k \alpha_m=\alpha$. For all n so large that $2\pi/(n+k-\alpha)<\min(\theta_{m+1}-\theta_m)$, where $\theta_{k+1}=2\pi+\theta_1$, we define numbers $t_j=t_j(n)$ and $\theta_m^*(n)$, $j=1,\ldots,n$ and $m=1,\ldots,k$, by the following procedure

$$t_1 = \theta_1+\frac{(2-\alpha_1)\pi}{n+k-\alpha},\ t_2 = t_1+\frac{2\pi}{n+k-\alpha},\ldots,\ t_{j_1} = t_{j_1-1}+\frac{2\pi}{n+k-\alpha},$$

where j_1 is determined by the inequality

$$t_{j_1} < \theta_2 \leqq t_{j_1}+\frac{2\pi}{n+k-\alpha},\ldots,\ t_{j_1+1} = t_{j_1}+\frac{2(2-\alpha_2)}{n+k-\alpha},$$

$$t_{j_1+2} = t_{j_1+1}+\frac{2\pi}{n+k-\alpha},\ldots,\ t_{j_2} = t_{j_2-1}+\frac{2\pi}{n+k-\alpha},$$

where j_2 is determined by the inequality $t_{j_2}<\theta_3\leqq t_{j_2}+2\pi/(n+k-\alpha)$;

$$\begin{aligned} t_{j_{k-1}+1} &= t_{j_{k-1}}+2(2-\alpha_k)\pi/(n+k-\alpha),\\ t_{j_{k-1}+2} &= t_{j_{k-1}+1}+2\pi/(n+k-\alpha),\ldots,\\ t_{j_k} &= t_{j_k-1}+2\pi/(n+k-\alpha); \end{aligned}$$

here j_k is determined by the inequality

$$t_{j_k} < \theta_1 + 2\pi \leqq t_{j_k}+2\pi/(n+k-\alpha);$$

finally,

$$\theta_1^* = \theta_1,\quad \theta_l^* = \tfrac{1}{2}(t_{j_{l-1}} + t_{j_{l-1}+1}),\qquad l = 2,\ldots,k.$$

Hence,

$$|\theta_m-\theta_m^*| \leqq \frac{(2-\alpha_m)\pi}{n+k-\alpha} < \frac{2\pi}{n} \quad \text{for } m = 1,\ldots,k. \tag{2.2}$$

We now define

$$P_{n+k}(z,k) = \prod_{j=1}^{n} (1-ze^{-it_j}) \prod_{m=1}^{k} (1-ze^{-i\theta_m^*}). \tag{2.3}$$

We shall prove

THEOREM 2.1. *For every k and every $z \in D$, $P_{n+k}(z, k) \to S_k(z)$, as $n \to \infty$.*

To prove this, we need a lemma of the Riemann-Lebesgue type. For each n, let s_n be the following step function on $[0, 2\pi]$. The jumps of s_n only occur at $t_1, \ldots, t_n$ and $\theta_1^*, \ldots, \theta_k^*$. Each jump at t_j is equal to 1 and each jump at θ_m^* is equal to $(1-\alpha_m)$. These conditions do not determine s_n completely; however, we find it more convenient to state the additional conditions in terms of the related function,

$$v_n(t) = s_n(t) - (n+k-\alpha)t/2\pi. \tag{2.4}$$

We set $v_n(\theta_1^*+) = \frac{1}{2}(1-\alpha_1)$ and define $v_n(\theta_m^*) = v_n(t_j) = 0$ for $m = 1, \ldots, k$ and $j = 1, \ldots, n$. Then we have the following

LEMMA 2.1. *If g is an integrable function on $(0, 2\pi)$, then*

$$\lim_{n\to\infty} \int_0^{2\pi} g(t) v_n(t)\, dt = 0. \tag{2.5}$$

The proof of this lemma is the same as the proof of the Riemann-Lebesgue lemma. We first prove it for the case of a step function, and then approximate g by step functions, cf. [9].

Now we can prove Theorem 2.1. For fixed $z \in D$, let g be the integrable function

$$g(t) = \frac{d}{dt} \log(1 - ze^{-it}) = \frac{d}{dt}\left\{-\sum_{p=1}^{\infty} \frac{z^p}{p} e^{-ipt}\right\}.$$

Here, as everywhere else in this paper, we choose the principal values of the logarithms. Integration by parts gives

$$\begin{aligned}\int_0^{2\pi} g(t) v_n(t)\, dt &= \int_0^{2\pi} -\log(1 - ze^{-it})\, dv_n(t) \\ &= -\sum_{j=1}^{n} \log(1 - ze^{-it_j}) - \sum_{m=1}^{k} (1-\alpha_m) \log(1 - ze^{-i\theta_m^*}),\end{aligned}$$

and this tends to 0 by the lemma. Using (2.2), we see that

$$\sum_{j=1}^{n} \log(1 - ze^{-it_j}) + \sum_{m=1}^{k} \log(1 - ze^{-i\theta_m^*}) \to \sum_{m=1}^{k} \alpha_m \log(1 - ze^{-i\theta_m^*}).$$

This is equivalent to $P_{n+k}(z, k) \to S_k(z)$ for each k and each $z \in D$.

3. Bounded approximation by $S_k(z)$.

The following lemma is a trivial consequence of Herglotz's theorem [4].

LEMMA 3.1. *Let f be holomorphic and zero free in D, $f(0) = 1$ and* $\sup |f(z)| = M < +\infty$. *Then there exists a nondecreasing real-valued function ν in* $[0, 2\pi]$, *with $\nu(0) = 0$ and $\nu(2\pi) = 2 \log M$, such that*

$$f(z) = \exp\left\{\int_0^{2\pi} \frac{-ze^{-it}}{1 - ze^{-it}}\, d\nu(t)\right\} \quad \textit{for each } z \textit{ in } D. \tag{3.1}$$

The above lemma indicates the importance of the following function,

$$f(z) = \exp\{-z/(1-z)\}$$

which is zero free, holomorphic and bounded in D. The first thing we do is to study the bounded approximation of $\exp\{-z/(1-z)\}$ by expressions of the form

$$S_k(z) = \prod_{m=1}^{k} (1 - ze^{-i2\pi m/k})^{\alpha_{k,m}} \tag{3.2}$$

throughout D, where $\alpha_{k,m} \geqq 0$ for $m=1,\ldots,k$ and

$$\max_{1\leqq m\leqq k} \alpha_{k,m} \to 0, \quad \text{as } k \to \infty. \tag{3.3}$$

We establish the following

THEOREM 3.1. *Let $f(z)=\exp\{-z/(1-z)\}$ and $\varepsilon>0$ be given. There exist functions S_k, $k=1,2,\ldots,$ of the form indicated in* (3.2) *and* (3.3), *such that S_k converges to f throughout D and*

$$\max_{|z|\leqq 1} |S_k(z)| \leqq e^{1/2+\varepsilon} \quad \textit{for all } k. \tag{3.4}$$

Proof. Let k and p be positive integers with $k>2p$. We define

$$\alpha_{k,m}^{(p)} = \sum_{j=1}^{p} \frac{2j}{k}\left(1-\frac{j}{p}\right)\left(1+\cos\frac{2\pi jm}{k}\right) \tag{3.5}$$

where $m=1,\ldots,k$. Clearly, each $\alpha_{k,m}^{(p)} \geqq 0$ and

$$\max_{1\leqq m\leqq k} \alpha_{k,m}^{(p)} \leqq \frac{p^2}{k}. \tag{3.6}$$

For each z in D,

$$\begin{aligned}
&\sum_{m=1}^{k}\left(1+\cos\frac{2\pi jm}{k}\right)\log(1-ze^{-i2\pi m/k}) \\
&\quad = -\sum_{\nu=1}^{\infty}\sum_{m=1}^{k}\frac{z^\nu}{\nu}e^{-i2\pi\nu m/k}\left(1+\cos\frac{2\pi jm}{k}\right) \\
&\quad = -\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu}\left\{\sum_{m=1}^{k}e^{-i2\pi\nu m/k}+\sum_{m=1}^{k}\cos\frac{2\pi\nu m}{k}\cdot\cos\frac{2\pi jm}{k}\right. \\
&\qquad\qquad\left. -i\sum_{m=1}^{k}\sin\frac{2\pi\nu m}{k}\cdot\cos\frac{2\pi jm}{k}\right\}.
\end{aligned} \tag{3.7}$$

Here,

$$\begin{aligned}
\sum_{m=1}^{k} e^{-i2\pi\nu m/k} &= k \quad \text{if } k\mid\nu, \\
&= 0 \quad \text{if } k\nmid\nu.
\end{aligned} \tag{3.8}$$

It follows that

$$\text{(3.9)}\quad \begin{aligned}\sum_{m=1}^{k}\cos\frac{2\pi\nu m}{k}\cdot\cos\frac{2\pi jm}{k} &= \frac{1}{2}\sum_{m=1}^{k}\left\{\cos\frac{2\pi(\nu+j)m}{k}+\cos\frac{2\pi(\nu-j)m}{k}\right\}\\ &= (1/2)k \quad \text{if } k|(\nu+j) \text{ or } k|(\nu-j),\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

(k cannot divide both $\nu+j$ and $\nu-j$ since $2j\leqq 2p\leqq k$.) Similarly,

$$\text{(3.10)}\quad \sum_{m=1}^{k}\sin\frac{2\pi\nu m}{k}\cos\frac{2\pi jm}{k} = \frac{1}{2}\sum_{m=1}^{k}\left\{\sin\frac{2\pi(\nu+j)m}{k}+\sin\frac{2\pi(\nu-j)m}{k}\right\} = 0.$$

Note that $k|(\nu+j)$ if and only if $\nu=\mu k-j$ where $\mu=1, 2,\ldots$ and $k|(\nu-j)$ if and only if $\nu=\mu k+j$ where $\mu=0, 1, 2,\ldots$. Hence, substituting (3.8), (3.9) and (3.10) into (3.7) and combining with (3.5), we get

$$\text{(3.11)}\quad \begin{aligned}\sum_{m=1}^{k}\alpha_{k,m}^{(p)}\log(1-ze^{-i2\pi m/k}) &= -\sum_{j=1}^{p}\frac{2j}{k}\left(1-\frac{j}{p}\right)\\ &\times\left\{\frac{k}{2}\cdot\frac{z^j}{j}+\sum_{\mu=1}^{\infty}\frac{z^{\mu k}}{\mu k}k+\sum_{\mu=1}^{\infty}\frac{z^{\mu k+j}}{\mu k+j}\cdot\frac{k}{2}+\sum_{\mu=1}^{\infty}\frac{z^{\mu k-j}}{\mu k-j}\cdot\frac{k}{2}\right\}.\end{aligned}$$

To estimate this expression, we compare the terms with denominators $\mu k\pm j$ with similar terms with denominators μk. For each z in D, we get, remembering $p<\frac{1}{2}k$,

$$\begin{aligned}&\left|\sum_{j=1}^{p}\frac{2j}{k}\left(1-\frac{j}{p}\right)\left\{\sum_{\mu=1}^{\infty}\frac{z^{\mu k\pm j}}{\mu k\pm j}\cdot\frac{k}{2}-\sum_{\mu=1}^{\infty}\frac{z^{\mu k}}{\mu k}\cdot\frac{k}{2}\right\}\right|\\ &\qquad\leqq\sum_{j=1}^{p}j\left(1-\frac{j}{p}\right)\sum_{\mu=1}^{\infty}\frac{j}{(\mu k-j)\mu k}\\ &\qquad\leqq\sum_{j=1}^{p}j^2\left(1-\frac{j}{p}\right)\sum_{\mu=1}^{\infty}\frac{2}{\mu^2k^2}<\frac{\pi^2}{36}\cdot\frac{p^3}{k^2}.\end{aligned}$$

Hence, uniformly for $z\in D$,

$$\text{(3.12)}\quad \begin{aligned}\sum_{m=1}^{k}\alpha_{k,m}^{(p)}\log(1-ze^{-i2\pi m/k}) &= -\sum_{j=1}^{p}\left(1-\frac{j}{p}\right)z^j+\sum_{j=1}^{p}\frac{2j}{k}\left(1-\frac{j}{p}\right)\\ &\quad\times\left\{-\sum_{\mu=1}^{\infty}\frac{z^{\mu k}}{\mu}-\frac{z^j}{2}\sum_{\mu=1}^{\infty}\frac{z^{\mu k}}{\mu}-\frac{z^{-j}}{2}\sum_{\mu=1}^{\infty}\frac{z^{\mu k}}{\mu}\right\}+O(p^3/k^2)\\ &= -\sum_{j=1}^{p}\left(1-\frac{j}{p}\right)z^j+\sum_{j=1}^{p}\frac{2j}{k}\left(1-\frac{j}{p}\right)\\ &\quad\times\left(1+\frac{z^j+z^{-j}}{2}\right)\log(1-z^k)+O(p^3/k^2).\end{aligned}$$

By the maximum principle for harmonic functions,

$$\begin{aligned}\max_{|z|\leqq 1} \sum_{m=1}^{k} \alpha_{k,m}^{(p)} \log |1-ze^{-i2\pi m/k}| &\leqq \max_{0\leqq\theta\leqq 2\pi} \left\{\sum_{j=1}^{p} -\left(1-\frac{j}{p}\right)\cos j\theta\right\} \\ &+ \max_{0\leqq\theta\leqq 2\pi} \left\{\sum_{j=1}^{p} \frac{2j}{k}\left(1-\frac{j}{p}\right)(1+\cos j\theta)\log|1-e^{ik\theta}|\right\} \\ &+O(p^3/k^2) = I_1+I_2+O(p^3/k^2),\end{aligned}$$

say. Since $\log|1-e^{ik\theta}| \leqq \log 2$,

$$I_2 \leqq 4\log 2 \sum_{j=1}^{p} \frac{j}{k}\left(1-\frac{j}{p}\right) < \frac{p^2}{k}.$$

Also, it is well known that

$$-\sum_{j=1}^{p}\left(1-\frac{j}{p}\right)\cos j\theta = \tfrac{1}{2}-F_p(\theta)$$

when $F_p(\theta)$ denotes the Fejér kernel of order p. Since the Fejér kernels are non-negative, we see that $I_1 \leqq \frac{1}{2}$ for all p. Hence,

$$\max_{|z|\leqq 1} \prod_{m=1}^{k} |1-ze^{-i2\pi m/k}|^{\alpha_{k,m}^{(p)}} \leqq \exp\left[\tfrac{1}{2}+O(p^2/k)\right]. \tag{3.13}$$

Furthermore, from (3.12) we see that for each $z \in D$,

$$\sum_{m=1}^{k} \alpha_{k,m}^{(p)} \log(1-ze^{-i2\pi m/k}) = \sum_{j=1}^{p} -\left(1-\frac{j}{p}\right)z^j + O\left(\frac{p^2}{k}|z|^{k-p}\right)+O\left(\frac{p^3}{k^2}\right).$$

We now let k tend to ∞ through the positive integers, and let p tend to ∞ in such a way that $p^2/k \to 0$. The series $-\sum_{j=1}^{\infty} z^j$ converges to $-z/(1-z)$ in D; it follows that the Cesàro means

$$-\sum_{j=1}^{p}\left(1-\frac{j}{p}\right)z^j$$

of the partial sums also converge to $-z/(1-z)$. Thus for $z \in D$,

$$\sum_{m=1}^{k} \alpha_{k,m}^{(p)} \log(1-ze^{-i2\pi m/k})$$

converges to $-z/(1-z)$. We conclude, by (3.6), that from the numbers $\alpha_{k,m}^{(p)}$, we can form a family of finite sequences $\{\alpha_{k,m}\}$, $m=1,\ldots,k$, $k=1,2,\ldots$ which satisfies the conditions of the theorem.

We are now ready to prove a more general result, namely

THEOREM 3.2. *Let f be any zero free holomorphic function in D, $f(0)=1$ and* $\sup|f(z)|=M<+\infty$. *Then for any $\varepsilon>0$, there exists a sequence of functions S_k of the form indicated in* (3.2) *and* (3.3), *such that S_k converges to f throughout D, and*

$$\max_{|z|\leqq 1}|S_k(z)| \leqq M^{1+\varepsilon} \quad \text{for all } k.$$

Proof. By Lemma 3.1, we can find a nondecreasing real-valued function $\nu(t)$ on $[0, 2\pi]$ with $\nu(0)=0$, $\nu(2\pi)=2\log M$ such that

$$f(z) = \exp\left\{\int_0^{2\pi} \frac{-ze^{-it}}{1-ze^{-it}}\,d\nu(t)\right\}$$

for $z\in D$. Thus, setting

$$\beta_{n,j} = \nu(2\pi j/n)-\nu(2\pi(j-1)/n),$$

where $j=1,\ldots,n$, so that $\beta_{n,j}\geqq 0$ and $\sum_{j=1}^{n}\beta_{n,j}=2\log M$, we have

$$f(z) = \lim_{n\to\infty} f_n(z),$$

where

$$f_n(z) = \prod_{j=1}^{n}\exp\left\{\beta_{n,j}\frac{-ze^{-i2\pi j/n}}{1-ze^{-i2\pi j/n}}\right\}.$$

It is clear that

$$\sup_{|z|<1}|f_n(z)| \leqq M \tag{3.14}$$

for all n. By Theorem 3.1, we can choose finite sequences $\gamma_{m,p}$, $p=1,\ldots,m$, $m=1,2,\ldots$, such that $\gamma_{m,p}\geqq 0$ and $\max_p \gamma_{m,p}\to 0$,

$$S_m(w) = \prod_{p=1}^{m}(1-we^{-i2\pi p/m})^{\gamma_{m,p}} \to \exp\left\{\frac{-w}{1-w}\right\}$$

for $w\in D$ and

$$\max_{|w|\leqq 1}|S_m(w)| \leqq e^{(1+\varepsilon)/2}$$

for all m. In particular, for fixed integers n, j and $z\in D$, the sequence

$$S_m(ze^{-i2\pi j/n})^{\beta_{n,j}}$$

converges to

$$\exp\{-\beta_{n,j}\,ze^{-i2\pi j/n}/(1-ze^{-i2\pi j/n})\}.$$

We now take our integers m of the form kn, $k=1,2,\ldots$, and define

$$S_k(z,n) = \prod_{j=1}^{n} S_{kn}(ze^{-i2\pi j/n})^{\beta_{n,j}} = \prod_{p=1}^{n}(1-ze^{-i2\pi p/n})^{\alpha_{kn,p}}$$

where

$$\alpha_{kn,p} = \sum_{j+q=p(\text{mod } kn)} \gamma_{kn,q}\beta_{n,j}$$

for $p=1,\ldots,kn$. Clearly, $\alpha_{kn,p}\geqq 0$ and

$$\max_p \alpha_{kn,p} \leqq \sum_{j=1}^{n} \beta_{n,j}\cdot\max_q \gamma_{kn,q} = 2\log M\cdot\max_q \gamma_{kn,q},$$

which tends to 0 as k tends to ∞. Furthermore, for fixed n and $z\in D$,

$$S_k(z,n)\to f_n(z), \quad \text{as } k\to\infty;$$

and for all k, and all n,

$$\max_{|z|\leqq 1} |S_k(z,n)| \leqq \prod_{j=1}^{n} \{e^{(1+\varepsilon)/2}\}^{\beta_{n,j}} = M^{1+\varepsilon}. \tag{3.15}$$

Now let $A_K=\{z : |z|\leqq 1-1/K\}$. By (3.14) and the Stieltjes-Osgood theorem, f_n converges uniformly to f on A_K; and by (3.15), and fixed n, $S_k(z,n)$ converges uniformly to f_n. Hence, for a fixed integer K, we determine $n=n_K$ so large that

$$\max_{z\in A_K} |f_n(z) - f(z)| < \frac{1}{K},$$

and determine $k=k_K$ so large that for $n=n_K$

$$\max_{z\in A_K} |S_k(z,n)-f_n(z)| < \frac{1}{K}.$$

Setting $S_K(z)=S_{k_K}(z,n_K)$, we complete the proof of the theorem.

4. **Bounded approximation by *C*-polynomials.** We observe the following

LEMMA 4.1. *For* $0<\alpha\leqq\pi/4$ *and* $|z|<1$, *we have*

$$\log|1-z|-\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\log|1-ze^{-it}|\,dt < 3. \tag{4.1}$$

Proof. By the maximum principle it is sufficient to consider $z=e^{i\theta}$, and by symmetry, we assume that $0<\theta\leqq\pi$. We first note that

$$\begin{aligned}\log|1-e^{i\theta}| -\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\log|1-e^{i(\theta-t)}|\,dt &= \log\sin\frac{\theta}{2}-\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\log\left|\sin\frac{\theta-t}{2}\right|dt\\ &\leqq \frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\log\left|\frac{\theta}{\theta-t}\right|dt+2\\ &= \frac{\theta}{2\alpha}\int_{-\alpha/\theta}^{\alpha/\theta}\log\frac{1}{|1-u|}\,du+2.\end{aligned}$$

The function

$$h(x) = \frac{1}{2x}\int_{-x}^{x}\log\frac{1}{|1-u|}\,du$$

is continuous on $(0,\infty)$, $h(0+)=0$, and $h(x)\to -\infty$ as $x\to+\infty$. Hence, $h(x)$ must be bounded above on $(0,\infty)$. Actually, it is bounded above by 1. This completes the proof of the lemma.

LEMMA 4.2. *For each* $m=2, 3, \ldots,$ *let* $\tau_k=\tau_k(m)$, $k=1, \ldots, m$, *be points on* $[0, 2\pi]$ *such that* $0<\tau_1<\cdots<\tau_m\leqq 2\pi$, *and, setting* $\tau_{m+1}=\tau_1+2\pi$,

$$\max_{1\leqq k\leqq m}(\tau_{k+1}-\tau_k)\Big/\min_{1\leqq k\leqq m}(\tau_{k+1}-\tau_k) \leqq A. \tag{4.2}$$

Also let $\omega_k=\omega_k(m)$ *and* $\alpha_k=\alpha_k(m)$ *be defined as*

$$\omega_k = \tfrac{1}{2}(\tau_{k+1}+\tau_k), \qquad \alpha_k = m(\tau_{k+1}-\tau_k)/2\pi.$$

Finally, let

$$R_m(z) = \prod_{k=1}^{m}(1-ze^{-i\omega_k})^{\alpha_k}, \qquad z\in D. \tag{4.3}$$

Then for all large m,

$$\max_{|z|\leqq 1}|R_m(z)| \leqq \exp(19A^3). \tag{4.4}$$

Proof. For convenience, set $\frac{1}{2}(\tau_{k+1}-\tau_k)=\Delta_k=\Delta_k(m)$, and let n_1 denote the positive integer $[\sqrt{2}A^2]+2$. Without loss of generality, we can assume that the maximum of $|R_m(z)|$ is attained at $e^{i\theta}$, where $\tau_m-2\pi\leqq\theta\leqq\tau_1$. Since

$$\int_0^{2\pi}\log|1-e^{it}|\,dt = 0,$$

we may write

$$\log|R_m(e^{i\theta})| = \sum_{k=1}^{m}\frac{m}{2\pi}\int_{-\Delta_k}^{\Delta_k}\log\left|\frac{1-e^{i(\theta-\omega_k)}}{1-e^{i(\theta-\omega_k-t)}}\right|dt.$$

We split the sum into two parts $\sum'+\sum''$, where $\sum'$ denotes the sum over $1\leqq k\leqq n_1-1$ and $m-n_1+1\leqq k\leqq m$, $\sum''$ denotes the sum over $n_1\leqq k\leqq m-n_1$.

By Lemma 4.1, we can conclude that for all large m (those for which $\Delta_k(m)\leqq\pi/4$),

$$\sum{}' = \sum{}'\alpha_k\left\{\frac{1}{2\Delta_k}\int_{-\Delta_k}^{\Delta_k}\log\left|\frac{1-e^{i(\theta-\omega_k)}}{1-e^{i(\theta-\omega_k-t)}}\right|dt\right\} \leqq 3\sum{}'\alpha_k.$$

By (4.2), we see that

$$\min_{1\leqq k\leqq m}(\tau_{k+1}-\tau_k) \geqq \frac{1}{A}\cdot\frac{2\pi}{m}; \quad \max_{1\leqq k\leqq m}(\tau_{k+1}-\tau_k) \leqq A\cdot\frac{2\pi}{m}.$$

Hence, since $A\geqq 1$,

$$\begin{aligned}\sum{}' &\leqq 3\cdot\max\alpha_k\cdot\text{number of terms in }\sum{}' \\ &\leqq 3\cdot\frac{m}{\pi}A\frac{2\pi}{m}(2n_1-1) < 18A^3.\end{aligned} \tag{4.5}$$

We now turn to $\sum''$. We can write

$$\begin{aligned}\sum{}'' &= \frac{m}{2\pi}\sum_{k=n_1}^{m-n_1}\int_0^{\Delta_k}\log\left|\frac{(1-e^{i(\theta-\omega_k)})^2}{(1-e^{i(\theta-\omega_k)})(1-e^{i(\theta-\omega_k+t)})}\right|dt \\ &= \frac{m}{2\pi}\sum_{k=n_1}^{m-n_1}\int_0^{\Delta_k}-\log\frac{(1-\cos(\theta-\omega_k-t))^{1/2}(1-\cos(\theta-\omega_k+t))^{1/2}}{1-\cos(\theta-\omega_k)}\,dt.\end{aligned}$$

Using the trigonometric identity

$$(1-\cos(\alpha-\beta))(1-\cos(\alpha+\beta)) = (\cos\alpha-\cos\beta)^2,$$

we obtain

$$\text{(4.6)}\quad \begin{aligned}\sum{}'' &= \frac{m}{2\pi}\sum_{k=n_1}^{m-n_1}\int_0^{\Delta_k} -\log\left(1-\frac{1-\cos t}{1-\cos(\theta-\omega_k)}\right)dt\\ &= \frac{m}{2\pi}\sum_{k=n_1}^{m-n_1}\int_0^{\Delta_k} -\log\left(1-\frac{\sin^2(t/2)}{\sin^2((\theta-\omega_k)/2)}\right)dt.\end{aligned}$$

Setting

$$B_1 = \{k : n_1 \leqq k \leqq m-n_1,\ 0 \leqq \omega_k-\theta \leqq \pi\},$$
$$B_2 = \{k : n_1 \leqq k \leqq n-n_1,\ \pi \leqq \omega_k-\theta \leqq 2\pi\},$$

we see that for $k \in B_1$,

$$\sin((\omega_k-\theta)/2) \geqq (\omega_k-\theta)/\pi,$$

and for $k \in B_2$,

$$\sin((\omega_k-\theta)/2) \geqq 2-(\omega_k-\theta)/\pi = [(2\pi+\theta)-\omega_k]/\pi.$$

Hence, for $0 \leqq t \leqq \Delta_k$ and $k \in B_1$,

$$0 \leqq \frac{\sin^2(t/2)}{\sin^2((\theta-\omega_k)/2)} \leqq \frac{(\Delta_k/2)^2}{[(\omega_k-\theta)/\pi]^2} \leqq \frac{\pi^2}{4}\frac{\Delta_k^2}{(\tau_k-\tau_1)^2}$$
$$\leqq \frac{\pi^2}{4^2}\cdot\frac{A^4}{(n_1-1)^2} < \frac{1}{2},$$

by the definition of n_1. The same inequality holds for $0 \leqq t \leqq \Delta_k$ and $k \in B_2$. Thus by the inequality

$$-\log(1-x) \leqq x/(1-x) \leqq 2x, \quad \text{for } 0 \leqq x \leqq \tfrac{1}{2},$$

and by (4.6), we get

$$\begin{aligned}\sum{}'' &\leqq \frac{m}{\pi}\sum_{k=n_1}^{m-n_1}\int_0^{\Delta_k}\frac{\sin^2(t/2)}{\sin^2((\theta-\theta_k)/2)}\,dt\\ &\leqq \frac{m}{\pi}\sum_{k\in B_1}\int_0^{\Delta_k}\frac{\pi^2}{4(\omega_k-\theta)^2}\,t^2\,dt\\ &\quad+\frac{m}{\pi}\sum_{k\in B_2}\int_0^{\Delta_k}\frac{\pi^2}{4[(2\pi+\theta)-\omega_k]^2}\,t^2\,dt\\ &\leqq \frac{\pi}{12}m\sum_{k\in B_1}\frac{\Delta_k^3}{(\tau_k-\tau_1)^2}+\frac{\pi}{12}m\sum_{k\in B_2}\frac{\Delta_k^3}{(\tau_m-\tau_{k+1})^2} < A^3.\end{aligned}$$

Combining this with (4.5), we obtain (4.4).

Now we have enough machinery to prove the main theorem, namely

THEOREM 4.1. *Let f be any zero free holomorphic function in D, $f(0)=1$ and* $\sup|f|=M<+\infty$. *Then there exists a sequence of C-polynomials*

$$P_n(z) = \prod_{j=1}^{n} (1-ze^{-it_{n,j}})$$

which converges to f in D, and is such that for an arbitrary $\varepsilon>0$,

$$\max_{|z|\leq 1} |P_n(z)| \leqq e^{20}\cdot M^{1+\varepsilon} \quad \textit{for all } n. \tag{4.7}$$

Proof. By Theorem 3.2, we can find a sequence of functions S_k of the form indicated in (3.2) and (3.3) which converges to f in D and is such that for all k,

$$\max_{|z|\leq 1} |S_k(z)| \leqq M^{1+\varepsilon}. \tag{4.8}$$

By (3.3) it may be assumed that for all k

$$\max_{1\leqq m\leqq k} a_{k,m} \leqq \varepsilon_1$$

where ε_1 is an arbitrarily fixed number between 0 and 1. By §2, we can find C-polynomials

$$P_{n+k}(z,k) = \prod_{j=1}^{n} (1-ze^{-it_j}) \prod_{m=1}^{k} (1-ze^{-i\theta_m^*}),$$

taking $\theta_m=2\pi m/k$ in §2, such that P_{n+k} converges to S_k for any fixed k.

To get a suitable bound of P_{n+k}, we will apply the above lemma, using the present t_j and θ_m^* as points ω_j. The intervals (τ_j, τ_{j+1}) in the lemma will have the points t_j and θ_m^* as midpoints. Thus, the interval (τ_j, τ_{j+1}) with midpoint t_j will have length $2\pi/(n+k-\alpha)$; the interval (τ_j, τ_{j+1}) with midpoint θ_m^* will have length $(1-\alpha_{k,m})2\pi/(n+k-\alpha)$. Since $p=n+k$, the exponent of $(1-z\exp(-it_j))$ in $R_p(z)=R_{n+k}(z)$ will be

$$\frac{n+k}{2\pi}\cdot\frac{2\pi}{n+k-\alpha} = \frac{n+k}{n+k-\alpha},$$

and the exponent of $(1-ze^{-i\theta_m^*})$ will be

$$\frac{n+k}{2\pi}(1-\alpha_{k,m})\frac{2\pi}{n+k-\alpha} = \frac{(1-\alpha_{k,m})(n+k)}{n+k-\alpha}.$$

It follows that we can take $A=1/(1-\varepsilon_1)$ in the lemma. Hence, for large n,

$$\max_{|z|\leq 1} |R_{n+k}(z)| \leqq \exp[19(1-\varepsilon_1)^{-3}].$$

That is,

$$\max_{|z|\leq 1} |P_{n+k}(z,k)| \leqq \exp[19(1-\varepsilon_1)^{-3}]\cdot \max_{|z|\leq 1} \prod_{m=1}^{k} |1-ze^{-i\theta_m^*}|^{\alpha_{k,m}}.$$

But $\theta_m^* \to \theta_m = 2\pi m/k$ and the maximum norm is continuous. Hence, for an arbitrary $\varepsilon_2 > 0$, we have, for large k,

$$\max_{|z|\leqq 1} \prod_{m=1}^{k} |1 - ze^{-i\theta_m^*}|^{\alpha_{k,m}} \leqq (1+\varepsilon_2) \max_{|z|\leqq 1} |S_k(z)|.$$

Taking ε_1 and ε_2 sufficiently small, we can conclude that for any k and sufficiently large n,

$$\max_{|z|\leqq 1} |P_{n+k}(z, k)| \leqq e^{20} M^{1+\varepsilon}.$$

Since S_k also satisfies a uniform boundedness condition (4.8), the same argument as in the proof of Theorem 3.2 implies that there is a sequence of C-polynomials

$$P_n(z) = P_{n+k}(z, k), \qquad n = n(k),$$

which satisfies (4.7) and which converges to f throughout D.

The bound in (4.7) can be improved just a little if we no longer require that the approximating polynomials have the value 1 at $z=0$. Indeed, let ε_n be a sequence of positive real numbers tending to 0; then there is a sequence of C-polynomials P_m satisfying (4.7) with ε replaced by ε_n. Multiplying P_n by $M^{-\varepsilon_n}$, we get a sequence of C-polynomials Q_n, which converges to f and satisfies

$$\max_{|z|\leqq 1} |Q_n(z)| \leqq e^{20} \cdot \sup_{|z|<1} |f(z)|. \tag{4.9}$$

That is, we obtain the following

COROLLARY 4.1. *A zero free bounded holomorphic function f in D can be boundedly approximated by C-polynomials Q_n in D, which satisfy the inequality* (4.9) *for all n.*

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